Variations on a Theorem of Landau. Part I

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1. Introduction. Recently [1] we discussed Landau's function $B(x)$ which equals the number of positive integers $\leqq x$ that can be expressed as a sum of two squares $u^2 + v^2$ with u and v nonnegative integers. We showed that

$$(1) \qquad B(x) = \frac{0.764223654\, x}{(\log x)^{1/2}}\left[1 + \frac{0.581948659}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right].$$

More generally, for any integer $n \neq -k^2$, let $B_n(x)$ be the number of positive integers $\leqq x$ of the form $u^2 + nv^2$. Then $B_1(x)$ is our previous $B(x)$, and it was also shown in [1] that

$$(2) \qquad B_4(x) = \frac{0.573167740\, x}{(\log x)^{1/2}}\left[1 + \frac{0.581948659}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right].$$

For the generalization $B_n(x)$ it is known that

$$(3) \qquad B_n(x) \sim \frac{b_n\, x}{(\log x)^{1/2}}$$

for some constant b_n, but to compute this constant is not always easy. As regards the error term in (3) H. H. Ostmann stated, in effect [2], that for $n > 0$,

$$(4) \qquad B_n(x) = \frac{b_n\, x}{(\log x)^{1/2}} + O\left(\frac{x}{\log x}\right),$$

but this is not always true. For example, it will be seen in the sequel that

$$(5) \qquad B_{14}(x) = \frac{b_{14}\, x}{(\log x)^{1/2}}\left[1 - \frac{\beta_{14}}{(\log x)^{1/4}} + O\left(\frac{1}{\log x}\right)\right]$$

for some constant β_{14}. On the other hand, for many values of n (namely, when there is one class/genus, as will be explained later) Eq. (4) can be strengthened to read

$$(6) \qquad B_n(x) = \frac{b_n\, x}{(\log x)^{1/2}}\left[1 + \frac{c_n}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right].$$

The erroneous (4) stems from a misinterpretation of the results obtained by R. D. James [3] and Gordon Pall [4]. They showed that, for a fixed *positive* n, the *total collection* of all positive numbers expressible by at least one form, $au^2 + buv + cv^2$ with $b^2 - 4ac = -4n$, has a population given by

$$(7) \qquad T_n(x) = \frac{t_n\, x}{(\log x)^{1/2}} + O\left(\frac{x}{\log x}\right)$$

for some t_n. If we have class number 1, all such quadratic forms are equivalent to $u^2 + nv^2$, and (4) follows. But if the class number exceeds unity, (4) does not

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automatically follow for the population of an *individual* form, and in (5), as was stated, the "error" is really of a higher order.

We shall not confine ourselves here either to class number 1 or to definite forms ($n > 0$). In the thesis of Paul Bernays [5], which is referred to in [6] and [7], but which was not available to us, he showed that for *any* a, b, and c such that $b^2 - 4ac = -4D \neq k^2$, the population of the individual form $au^2 + buv + cv^2$ satisfies

$$P(x) \sim \frac{p_D\, x}{(\log x)^{1/2}} \tag{8}$$

for some p_D. From this, therefore, one not only has (3), but also the further fact that all forms with the same determinant D have populations that are asymptotic to one another.

If one knew the constant t_n of (7), and if the several classes were always disjoint, then one could obtain the b_n of (3) merely by dividing t_n by the class number. But the classes are not always disjoint and the constants are generally not known. Further, the great variety of second-order terms, as in (5) and (6), has not been investigated. We propose to examine these questions here.

For many values of n, with various class numbers, or various group structures with the same class number, and with different types of overlap amongst the several classes, we will evaluate the constants b_n, indicate the nature of the second-order term, tabulate the exact populations $B_n(x)$, and compare the populations of the different classes.

We thought, at first, to have a subtitle: "Studies in Binary Quadratic Forms" for we must admit that the variety of situations that presented themselves far exceeded our original notions, and required some effort on our part to unravel and understand. Later, we considered whether another alteration of the title was called for, since the investigation, as it developed, was a stimulating one which threw off suggestions in many directions.

Some of these suggestions (only some!) relate to the generalized Riemann Hypothesis, to the exhibition of infinitely many examples having the same class number, to a criterion for Mersenne primes, to a problem of Bateman, to nonalgebraic singularities that can be dominated by algebraic singularities, to trees of prime classes, to the Dirichlet arithmetic progression theorem, and to the failure of factorization using idoneal number techniques. We will present some of these topics as we proceed, interspersing them within the main framework that is called for by the investigation of $B_n(x)$.

2. Notation. We have defined $B_n(x)$. Similarly, let $B_{a,c}(x)$ equal the population of $au^2 + cv^2$; we suppress the first subscript if it equals 1. Further, let $B_{a,b,c}(x)$ be the population of $au^2 + buv + cv^2$. And in this last case we must allow u or v to be negative; e.g., $37 = 4u^2 + 4uv + 5v^2$ for $(u, v) = (-1, 3)$, but 37 is not obtained if the arguments are nonnegative.

Sometimes we are especially interested in the odd (or even) numbers represented by these forms, and we then replace B by O (or E). Thus $O_{2,3}(x)$ is the number of positive odd integers $\leqq x$ represented by $2u^2 + 3v^2$. In [1] it was shown that if we have $O_1(x)$, this suffices, and we can easily deduce $E_1(x)$ and $B_1(x)$. One of our tasks here is to determine the extent to which this generalizes for $O_n(x)$.

We defined b_n in (3) and we defined c_n in (6) when (6) is valid.

We will have much need of the Dirichlet series $L_n(s)$ defined by

$$L_n(s) = \sum_{k=0}^{\infty} \left(\frac{-n}{2k+1}\right)(2k+1)^{-s} \tag{9}$$

and its analytic continuation, where $(-n/(2k+1))$ is the Jacobi symbol, cf. [8]. The function $\zeta_n(s)$ is given by

$$\zeta_n(s) = \prod_{r \mid 2n} \left(1 - \frac{1}{r^s}\right)\zeta(s) \tag{10}$$

where the product is taken over all primes r that divide $2n$, and where $\zeta(s)$ is the Riemann zeta function, cf. [9]. This function $\zeta_n(s)$ can always be written as an L function since it equals $L_{-m^2}(s)$ where m is the product of all distinct odd primes dividing n. Whichever notation is more convenient will be used in a particular instance.

The quantities g_n are also needed. Here

$$g_n = \prod_q (1 - q^{-2})^{-1/2} \tag{11}$$

where the q's are all the odd primes that have $-n$ as a quadratic nonresidue:

$$(-n/q) = -1.$$

If the primes p satisfy

$$(-n/p) = +1,$$

then

$$\zeta_n(s) = \prod_{p,q} 1/(1 - 1/p^s)(1 - 1/q^s) \tag{12}$$

while

$$L_n(s) = \prod_{p,q} 1/(1 - 1/p^s)(1 + 1/q^s). \tag{13}$$

Therefore

$$\frac{\zeta_n(s)}{L_n(s)} = \prod_q \frac{1 + 1/q^s}{1 - 1/q^s} = \prod_q \frac{1 - 1/q^{2s}}{(1 - 1/q^s)^2} \tag{14}$$

or

$$\prod_q \frac{1}{1 - 1/q^s} = \left(\frac{\zeta_n(s)}{L_n(s)}\right)^{1/2} \prod_q \frac{1}{(1 - 1/q^{2s})^{1/2}}.$$

It follows, by induction, that g_n may be evaluated by the very rapidly convergent product

$$g_n = \left(\frac{\zeta_n(2)}{L_n(2)}\right)^{1/4} \left(\frac{\zeta_n(4)}{L_n(4)}\right)^{1/8} \left(\frac{\zeta_n(8)}{L_n(8)}\right)^{1/16} \cdots \tag{15}$$

provided that the constants $L_n(2^k)$ are available. The formula in (15) may be nested:

$$g_n = \sqrt[4]{\{[\zeta_n(2)/L_n(2)]\sqrt{\{[\zeta_n(4)/L_n(4)]\sqrt{\{\cdots\}}\}}\}},$$

and for the accuracies aimed at here three terms generally suffice.

We also need the generating functions $f_n(s)$ (or $f_{a,c}(s)$, or $f_{a,b,c}(s)$) where

$$f_n(s) = \sum_{k=1}^{\infty} a_k k^{-s} \tag{16}$$

with $a_k = 1$ or 0 according as k is, or is not, representable by $u^2 + nv^2$, (or $au^2 + cv^2$, etc.). For example, cf. [1, p. 81],

$$f_1(s) = \frac{1}{1 - 1/2^s} \prod_p \frac{1}{1 - 1/p^s} \prod_q \frac{1}{1 - 1/q^{2s}} \tag{17}$$

where the primes $p \equiv 1 \pmod 4$ and the primes $q \equiv -1 \pmod 4$.

If n corresponds to a large class number, or to many genera, or if it has square factors, $N^2 \mid n$, or if we have some combination of these, $f_n(s)$ may be much more complicated. This creates one of our main tasks.

It will be convenient to introduce an abbreviated notation. Let $[P]$ be given by

$$[P] = \prod_p \frac{1}{1 - 1/p^s}, \tag{18}$$

the product being taken over some class of primes P. Further let [2], say, or $[-P]$, or $[Q^2]$ be given by

$$\frac{1}{1 - 1/2^s}, \qquad \prod_p \frac{1}{1 + 1/p^s}, \qquad \prod_q \frac{1}{1 - 1/q^{2s}}$$

respectively. A subscript, such as in $[Q]_2$, will mean that $s = 2$. Thus (11) and (17) may be written

$$g_n = ([Q]_2)^{1/2} \tag{19}$$

and

$$f_1(s) = [2][P][Q^2]. \tag{20}$$

In what follows, whenever we are discussing $B_n(x)$, by P we mean the class of primes p that have $-n$ as a quadratic residue, while Q is the class of primes q that have $-n$ as a quadratic nonresidue:

$$(-n/p) = +1, \qquad (-n/q) = -1.$$

3. Class Number 1. We start with the easier cases. For $n = 1$, a number m can be expressed as $m = u^2 + v^2$ if and only if in its factorization into primes, namely

$$m = 2^\alpha \prod p_i^{\alpha_i} \prod q_i^{2\beta_i},$$

we find an arbitrary power of 2, $\alpha \geqq 0$, arbitrary powers of any number of primes $p_i \equiv 1 \pmod 4$ and arbitrary *even* powers of any number of primes $q_i \equiv -1 \pmod 4$. Our abbreviated expression for the generator $f_1(s)$, in (20), indicates this symbolically.

Suppose for some n every p in P can be expressed as $u^2 + nv^2$; that is, assume that

$$(-n/p) = 1 \quad \text{implies} \quad p = u^2 + nv^2.$$

Then we also have class number 1, as in the case $n = 1$. This occurs for $n = 7, 4, 3, 2, 1, -2, -5, -13, -17, -29, -41, -53, -61, -73, -89, -97$ and some larger negative primes. In our choice of examples, however, we will largely confine ourselves, here and later, to the 200 values of $n \neq -k^2$ that satisfy $|n| \leqq 105$. Naturally, we will not examine in detail even all of these; in fact, some of them are quite complicated.

The 16 class number 1 cases mentioned above have four different types of behavior.

(a) $n = -2, 1, 2$.

Here $u^2 + nv^2$ not only represents every prime p in P but also represents the prime 2.

(b) $n = -61, -53, -29, -13, -5, 3$.

Here n can be certain positive or negative primes $\equiv 3 \pmod 8$. For some other such primes, e.g., $n = -101, -37, +11, +19$, etc. the class number is 3 or larger. Some of those cases are examined later. In this case (b) 2^α is represented by $u^2 + nv^2$ if and only if α is even. In other words, the prime 2 acts like a prime q here in distinction to its behavior in case (a) where it acts like a prime p. But the remaining prime $|n|$ is again representable, and thus behaves like a p.

(c) $n = -97, -89, -73, -41, -17, 7$.

Here n can be certain positive or negative primes $\equiv 7 \pmod 8$, but again not every such prime since $-257, +23, +31$, etc. have class number 3 or larger. This time $2^\alpha = u^2 + nv^2$ if and only if $\alpha \geqq 2$. While 2 is not representable, 8 is. Specifically:

$$8 = 1^2 + 7\cdot 1^2 = 5^2 - 17\cdot 1^2 = 7^2 - 41\cdot 1^2 = 9^2 - 73\cdot 1^2$$
$$= 217^2 - 89\cdot 23^2 = 69^2 - 97\cdot 7^2.$$

The above assertion follows since 4 is obviously representable as $4 = 2^2 + n\cdot 0^2$, and all higher powers of 2 can be written as 4^k or $8\cdot 4^k$. Again, $|n|$ is representable.

(d) $n = 4$.

Here, as in (c), 2^α is representable if and only if $\alpha \geqq 2$, but the main difference here is that n is now divisible by a square > 1. These cases will be treated separately as we proceed.

It follows that the generating functions in these four cases are given by the following four formulas.

$$\text{(22)}\quad \begin{aligned} &\text{(a)} && f_n(s) = [2][P][Q^2],\\ &\text{(b)} && f_n(s) = [4][|n|][P][Q^2],\\ &\text{(c)} && f_n(s) = (1 - 1/2^s + 1/4^s)[2][|n|][P][Q^2],\\ &\text{(d)} && f_4(s) = (1 - 1/2^s + 1/4^s)[2][P][Q^2]. \end{aligned}$$

Now, for any n, we have

$$\text{(23)}\qquad \zeta_n(s)L_n(s) = [P]^2[Q^2]$$

from (12) and (13). Therefore, the common factor $[P][Q^2]$ in (22) may be written

$$[P][Q^2] = (\zeta_n(s)L_n(s))^{1/2}([Q^2])^{1/2},$$

and using (10) we have

$$(24) \qquad [P][Q^2] = \prod_{r|2n} (1 - 1/r^s)^{1/2}(\zeta(s)L_n(s))^{1/2}([Q^2])^{1/2}.$$

Since $\zeta(s)$ has the principal part: $1/(s - 1)$, at $s = 1$, and the remaining factors in (24) and (22) are analytic at $s = 1$, any one of the generating functions in (22) may be expanded as follows.

$$(25) \qquad f_n(s) = D_n/(s - 1)^{1/2}[1 + \alpha_{1,n}(s - 1) + \alpha_{2,n}(s - 1)^2 + \cdots]$$

where the constant D_n has the common factor

$$\prod_{r|2n} (1 - 1/r)^{1/2}(L_n(1))^{1/2}g_n$$

from (24) and (11), and this is to be multiplied by

$$2, \quad \tfrac{4}{3}|n|/(|n| - 1), \quad \tfrac{3}{2}|n|/(|n| - 1), \quad \text{or} \quad \tfrac{3}{2},$$

depending on whether we have case (a), (b), (c), or (d), respectively.

Alternatively, we may use Euler's ϕ function and may verify that

$$(26) \qquad D_n = \delta_n[L_n(1) \cdot 2|n|/\phi(2|n|)]^{1/2}g_n$$

where δ_n is given by

$$\delta_n = 1, \tfrac{2}{3}, \tfrac{3}{4}, \text{or } \tfrac{3}{4}$$

respectively.

From the generator (25) the analysis of $B_n(x)$ in these cases proceeds almost exactly as in Landau's original paper on $B_1(x)$ (see [10], or cf. [1]), the main difference being that the $L(s) = L_1(s)$ there is now replaced by $L_n(s)$. One, therefore, also obtains equation (6) in these cases, but with the coefficient now given by

$$(27) \qquad b_n = D_n/\Gamma(\tfrac{1}{2}) = \delta_n g_n[L_n(1) \cdot 2|n|/\pi\phi(2|n|)]^{1/2}.$$

The constants c_n in (6) are more difficult to compute; we will not calculate them here, but for $n > 0$ a computation analogous to that for c_1 in [1] is possible since $L_n'(1)/L_n(1)$ can be evaluated in terms of gamma functions when n is positive.

4. Calculation of b_n. First Digression. Since $L_n(1)$ is expressible in closed form, cf. [8], the only problem in evaluating b_n from (27) is that of computing the quantity g_n. The product in (11) converges very poorly, and *if one must depend upon this* an accurate value of b_n is not obtainable. We will see later that this is the case for some interesting examples: $n = 11, 14, 21$, etc. But if the numbers $L_n(2^k)$ are available, say, for $k = 1, 2, 3$, the very rapidly convergent (15) may be used.

Now, for any $n < 0$, $L_n(2m)$ is available in closed form (at least in principle) for all even integers $2m$, [8]. For any *negative* n, therefore, (15) enables us to compute g_n accurately. For some positive values of n, including $1 \leqq n \leqq 10$, the $L_n(2m)$ have been computed numerically, [8], [11], and (15) again may be used. It follows that in all our 16 class number 1 cases, we could compute b_n accurately from (27) and (15), although for the larger values of n, such as -97 and -89, the computation would be tedious. We include within a larger Table 1 accurate values of b_n for $n = -17, -5, -2, 1, 2, 3, 4$, and 7 selected from the 16 cases presently under consideration.

TABLE 1
Constants

$b_{-34} = 0.505360417$	$b_2 = 0.872887558$	$b_{12} = 0.399318378$
$b_{-17} = 0.484644756$	$b_3 = 0.638909405$	$b_{13} \approx 0.420(6)$
$b_{-10} = 0.488162034$	$b_4 = 0.573167740$	$b_{14} \approx 0.563(5)$
$b_{-8} = 0.344664285$	$b_5 = 0.535179999$	$b_{16} = 0.334347848$
$b_{-7} = 0.455065213$	$b_6 = 0.558357114$	$b_{20} = 0.401384999$
$b_{-6} = 0.482889041$	$b_7 = 0.543539641$	$b_{24} = 0.279178557$
$b_{-5} = 0.515939482$	$b_8 = 0.436443779$	$b_{27} = 0.496929538$
$b_{-3} = 0.441875842$	$b_9 = 0.424568696$	$b_{64} = 0.274642876$
$b_{-2} = 0.689328571$	$b_{10} = 0.473558100$	$b_{96} = 0.209383918$
$b_1 = 0.764223654$	$b_{11} \approx 0.677(4)$	$b_{256} = 0.259716632$

We illustrate the foregoing discussion by a computation of b_{-17}, a number that we will need later anyway. We have, from (27),

$$b_{-17} = \tfrac{3}{4}g_{-17}(34L_{-17}(1)/16\pi)^{1/2}.$$

Here we want the closed formula [8]:

$$L_{-17}(1) = 17^{-1/2}\log(4 + 17^{1/2}) = 17^{-1/2}\log\left[8 + \frac{1}{8+}\,\frac{1}{8+}\cdots\right]$$
$$= 0.5080424169.$$

To compute g_n for negative n in the manner indicated above we note that, for negative n, $\zeta_n(2k)$ and $L_n(2k)$ can be expressed [8] as:

$$\zeta_n(2k) = r_{n,k}\pi^{2k}, \qquad L_n(2k) = s_{n,k}\pi^{2k}/(-n)^{1/2} \tag{28}$$

for computable rational numbers $r_{n,k}$ and $s_{n,k}$. Thus, in our case, we obtain

$$\frac{\zeta_{-17}(2)}{L_{-17}(2)} = \frac{6}{\sqrt{17}}, \qquad \frac{\zeta_{-17}(4)}{L_{-17}(4)} = \frac{174}{41\sqrt{17}}, \qquad \frac{\zeta_{-17}(8)}{L_{-17}(8)} = \frac{123529038}{29950897\sqrt{17}},$$

and therefore, from the rapidly convergent product (15), we find

$$g_{-17} = 1.102320127.$$

Finally we compute

$$b_{-17} = 0.484644756.$$

We digress here to note that the Eqs. (28) can be used to give a simple, attractive proof of a known

THEOREM. *If $a > 0$ and $\neq k^2$ then a is a quadratic residue for infinitely many primes and a quadratic nonresidue for infinitely many.*

Proof. For such an a, from (12), (13), and (28), we have

$$\frac{\zeta_{-a}(2)}{L_{-a}(2)} = \prod_q \frac{q^2+1}{q^2-1} = k_a a^{1/2}$$

and

$$\frac{\zeta_{-a}(2)L_{-a}(2)}{\zeta_{-a}(4)} = \prod_p \frac{p^2+1}{p^2-1} = l_a a^{1/2}$$

for rational numbers k_a and l_a, with the products taken over all primes q and p such that $(a \mid q) = -1$ and $(a \mid p) = +1$. If the theorem is assumed false we obtain a contradiction, since for these values of a, $a^{1/2}$ is irrational.

Returning to b_n, we also note that it is easier to compute a *rough* value of b_n for $n > 0$ than for $n < 0$, but much more difficult to compute an *accurate* value when $n > 0$. This is because the $L_n(1)$ are simpler if $n > 0$, e.g., $L_7(1) = \pi/2\sqrt{7}$, and, for all n, g_n is larger than, but close to, unity. Thus b_7 is bounded below by the simply computed $(252)^{1/4}/8 = 0.498$. On the other hand, for $n > 0$ the $L_n(2^k)$ are not available in closed form, and therefore g_n is *much* harder to compute accurately.

We might note that while James [3] and Pall [4] confined themselves to definite forms, in this matter of the accurate computation of the b_n, the indefinite forms are easier. Further, as already stressed by Gauss [15], when $n < 0$ we encounter many more examples of small class number, and of small number of classes/genus, both of which conditions greatly simplify the theory. We already noted above more class number 1 cases with $n < 0$, and for other relatively simple cases this predominance becomes even greater.

5. Calculation of B_n. Some Complications. The exact values of $B_n(x)$ given in the tables here were obtained by the second-named author with an IBM 7090. For the values of n examined the average program ran 10 minutes.

In the efficient computation of $B_1(x)$ discussed in [1, p. 81] we merely counted the population of the odd numbers, $O_1(x)$, and deduced the values of $E_1(x)$ and $B_1(x)$ by the simple recurrence relations:

$$E_1(x) = E_1(x/2) + O_1(x/2),$$
$$B_1(x) = E_1(x) + O_1(x). \tag{29}$$

Since these odd numbers are given by $m^2 + 4M^2$, where m goes through the *odd* integers and M goes through *all* integers, one successively adds $1 + 8 + 16 + 24 + \cdots$ to a fixed $4M^2$ and records, by a mark, all such sums. It develops, therefore, that the program consists almost exclusively of counting and "logical" operations.

The generalization of that program used here for $B_n(x)$ computed $Am^2 + BM^2$ for fixed integers A and B and for m odd, as before. For $B_2(x)$ everything proceeds as before by computing $m^2 + 2M^2$. But for $B_3(x)$ two modifications are necessary.

Firstly, there are *two* types of odd numbers of the form $u^2 + 3v^2$, those with v odd, and those with v even:

$$3m^2 + 4M^2 \quad \text{and} \quad m^2 + 12M^2.$$

As may be seen, these two sets are disjoint so that we have

$$O_3(x) = O_{3,4}(x) + O_{12}(x).$$

One therefore computes the two counts on the right separately, and then merely adds. A similar problem arises for any $B_n(x)$ with odd n, but while a similar solution is available for, say, $n = 7$, with

$$O_7(x) = O_{7,4}(x) + O_{28}(x),$$

for $n = 5$ this device fails, since

$$5m^2 + 4M^2 \quad \text{and} \quad m^2 + 20M^2$$

are *not* disjoint, and have instead a complicated intersection. This mysterious intersection becomes clarified only after we have analyzed the cyclic class number 4 cases. Similarly, while one immediately observes that the counts $O_{3,4}(x)$ and $O_{12}(x)$ are nearly equal, this only becomes understandable after we study class number 2. The desire to attain greater computing efficiency here, by counting only the odd numbers, therefore forces us to examine these interesting phenomena.

A second modification needed in computing some $B_n(x)$ in this way stems from the fact (page 555) that factors of 2^α may occur according to different laws, and consequently that the recurrence in (29) may need to be altered. For $n = 3$, or generally for type (b) of page 555, one must replace the first line of (29) by

$$E_n(x) = E_n(x/4) + O_n(x/4), \tag{29a}$$

while for $n = 4$, or generally for types (c) or (d), one uses

$$E_n(x) = E_n(x/2) + O_n(x/4). \tag{29b}$$

Still another type of recurrence is needed for certain class number 2 cases such as $n = 6$, but we postpone that until later.

A third complication in computing $B_n(x)$ occurs for all negative n. Writing $n = -N$ temporarily we consider the indefinite form:

$$u^2 - Nv^2 = m. \tag{30}$$

Our previous procedure, of determining (in effect) *all* solutions of $u^2 + nv^2 = m$ for all (positive, odd) values of $m \leqq x$ is no longer feasible. For now if m has any solution (u, v) in (30), it has infinitely many solutions. The algorithm may be kept finite, however, by the use of Theorem 108 in Nagell [12]:

If $u + vN^{1/2}$ *is a fundamental solution of* (30) *and* $x + yN^{1/2}$ *is the fundamental solution of*

$$x^2 - Ny^2 = 1, \tag{31}$$

then

$$\begin{aligned} 0 \leqq v &\leqq m^{1/2}\, y/(2(x+1))^{1/2}, \\ 0 < |u| &\leqq [(x+1)m/2]^{1/2}. \end{aligned} \tag{32}$$

For any m it follows that we can restrict the variable v to those satisfying

$$v^2 \leqq my^2/2(x+1),$$

and, for any fixed v, we can insist upon

$$u \geqq v(x+1)/y. \tag{33}$$

For example, since the fundamental solutions of (31) for $N = 2$ and $N = 34$ are $3 + 2\sqrt{2}$ and $35 + 6\sqrt{34}$ respectively, we can confine ourselves to $v^2 \leqq m/2$, together with

$$u \geqq 2v \quad \text{and} \quad u \geqq 6v$$

in these two cases.

TABLE 2
Populations, Class No. 1

x	$B_1(x)$	$B_2(x)$	$B_3(x)$	$B_4(x)$	$B_{-2}(x)$
2^0	1	1	1	1	1
2^1	2	2	1	1	2
2^2	3	4	3	2	3
2^3	5	6	4	4	5
2^4	9	10	8	7	8
2^5	16	18	14	12	15
2^6	29	33	25	22	26
2^7	54	60	45	41	48
2^8	97	111	82	72	87
2^9	180	205	151	137	161
2^{10}	337	385	282	254	299
2^{11}	633	725	531	476	563
2^{12}	1197	1374	1003	901	1066
2^{13}	2280	2610	1907	1716	2030
2^{14}	4357	4993	3645	3274	3885
2^{15}	8363	9578	6993	6286	7464
2^{16}	16096	18426	13456	12090	14384
2^{17}	31064	35568	25978	23331	27779
2^{18}	60108	68806	50248	45140	53782
2^{19}	116555	133411	97446	87511	104359
2^{20}	226419	259145	189291	169972	202838
2^{21}	440616	504222	368338	330752	394860
2^{22}	858696	982538	717804	644499	769777
2^{23}	1675603	1917190	1400699	1257523	1502603
2^{24}	3273643	3745385	2736534	2456736	2936519
2^{25}	6402706	7324822	5352182	4804666	5744932
2^{26}	12534812		10478044	9405749	

We note, then, that the fundamental solution of (31) is needed not only in evaluating the number $L_{-n}(1)$ (which is used in computing the constant b_{-n}) but it is also needed here in computing $B_{-n}(x)$.

6. First Table of $B_n(x)$. Comparisons. In Table 2 we list values of $B_n(x)$ for five class number 1 cases, and for arguments $x = 2^k$ up to $x = 2^{25} = 33554432$ or $x = 2^{26} = 67108864$. An interested reader can easily recover the corresponding values of $O_n(x)$ and $E_n(x)$, if he wishes, by use of the recurrences (29, 29a, 29b).

From (6) we have

$$\frac{B_n(x)}{B_1(x)} = \frac{b_n}{b_1}\left[1 + \frac{c_n - c_1}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right]. \tag{34}$$

One finds from Tables 1 and 2 that $B_2(x)/B_1(x)$ approaches b_2/b_1 from *above*, while $B_{-2}(x)/B_1(x)$ approaches b_{-2}/b_1 from *below*. This implies that

$$c_2 > c_1 = 0.581948659 > c_{-2}$$

but we have not computed $c_{\pm 2}$ to verify this. $B_3(x)$ remains so closely proportional to $B_1(x)$ that it is not clear from this data whether $c_3 > c_1$ or $c_3 < c_1$. It would be unlikely that they are exactly equal. But $c_4 = c_1$, as is known [1, Theorem 2]. In fact, we have the theoretical relation

$$\frac{B_4(x)}{B_1(x)} = \frac{3}{4}\left[1 + \frac{1/4}{(\log_2 x)^2} + \frac{1.46457444}{(\log_2 x)^3} + O\left(\frac{1}{(\log_2 x)^4}\right)\right].$$

The column $B_4(x)$ in Table 2 is actually redundant there since one has

$$B_4(x) = B_1(x) - B_1(x/2) + B_1(x/4).$$

We note, in passing, that of all binary forms $u^2 + nv^2$, $u^2 + 2v^2$ is the most populous, since b_2 is the largest of these constants. Similarly, it is of interest that $u^2 + v^2 + 2w^2$ appears to be the most populous of all ternary forms $u^2 + v^2 + nw^2$. However, there is no exact correlation here since $u^2 + v^2 + nw^2$ represents 20/24, 22/24, 21/24, 17/24, 20/24, or 21/24 of all numbers when n equals 1 to 6, respectively, and this is not quite in the same order as the magnitudes of b_n. The theory appears to be incomplete; e.g., for $n = 10$ the fraction is unknown [13].

7. Class Number 2 and the Generalized Riemann Hypothesis. There are 64 class number 2 cases in the range $-105 \leqq n \leqq 105$. All of these are characterized by the fact that the primes p in P fall into two (equinumerous) classes, those in a class P_0 that *are* representable by the form $F_0 : u^2 + nv^2$, and those in a second class P_1 *not* representable by F_0, but which are representable by another form $F_1 : au^2 + buv + cv^2$ with $b^2 - 4ac = -4n$. Consider first the 32 cases where $\pm n$ is a prime, π, or twice a prime, 2π. The behavior here is analogous to the class number 1 type (a) in Section 3, but already there are more variations. We may have different formulas for F_1 and differing ways of representing the prime divisors of $2n$. If such a prime divisor is representable by F_0 we say it is in P_0', while if it is representable by F_1 we say it is in P_1'. One, and only one, of these representations is possible. In Tables 3 and 4 we show the variations that occur. Thus, for $n = 5$, 5 is representable by $u^2 + 5v^2$ (obviously), and 2 is representable by $2u^2 + 2uv + 3v^2$ (also obviously).

For all class number 2 cases let the two classes of *all* positive numbers representable by F_0 and F_1 be Φ_0 and Φ_1. Under *composition* these two classes satisfy the group multiplication of the group of order two:

$$\Phi_i\Phi_j = \Phi_k$$

with

$$k \equiv i + j \pmod 2.$$

TABLE 3
Variations, Class No. 2

Type	Values of n
a_1	5, 13, 37
a_2	−86, −38, −22, −6
a_3	−83, −67, −59, −43, −19, −11, −3
a_4	6, 10, 22, 58
a_5	−74, −58, −26, −10
a_6	−103, −71, −47, −31, −23, −7
a_7	−94, −62, −46, −14

TABLE 4
Behavior in the Different Types

Type	Form of n	F_0	F_1	P_0'	P_1'
a_1	π	$u^2 + \pi v^2$	$2u^2 + 2uv + \left(\frac{\pi + 1}{2}\right)v^2$	π	2
a_2	-2π	$u^2 - 2\pi v^2$	$2\pi u^2 - v^2$	π	2
a_3	$-\pi$	$u^2 - \pi v^2$	$\pi u^2 - v^2$	—	2, π
a_4	2π	$u^2 + 2\pi v^2$	$2u^2 + \pi v^2$	—	2, π
a_5	-2π	$u^2 - 2\pi v^2$	$2u^2 - \pi v^2$	—	2, π
a_6	$-\pi$	$u^2 - \pi v^2$	$\pi u^2 - v^2$	2	π
a_7	-2π	$u^2 - 2\pi v^2$	$2\pi u^2 - v^2$	2	π

It follows that a positive number prime to $2n$ is in Φ_0 or Φ_1, according as the total number of its prime factors which are contained in P_1 (counting multiplicity) is even or odd. Therefore the subsets of the numbers in Φ_0 and Φ_1 prime to $2n$ have generating functions:

$$\text{(35)}\qquad \begin{aligned} &\tfrac{1}{2}[P_0][Q^2]\{[P_1] + [-P_1]\}, \\ &\tfrac{1}{2}[P_0][Q^2]\{[P_1] - [-P_1]\} \end{aligned}$$

for Φ_0 and Φ_1 respectively. The generating functions for *all* members of these sets are obtained from (35) simply by including factors for those primes in P_0' or P_1' alongside the corresponding factors $[P_0]$ or $[P_1]$. Thus, for $n = 5$, type a_1, we have the generators

$$\tfrac{1}{2}[P_0][5][Q^2]\{[2][P_1] \pm [-2][-P_1]\}$$

where we take the $+$ sign for $f_5(s)$ and the $-$ sign for $f_{2,2,3}(s)$. Similarly, for $n = 6$, type a_4, we have

$$\text{(36)}\qquad \tfrac{1}{2}[P_0][Q^2]\{[2][3][P_1] \pm [-2][-3][-P_1]\}$$

for $f_6(s)$ and $f_{2,3}(s)$ respectively.

In all these generators the function $[P] = [P_0][P_1]$ has the same type of singularity at $s = 1$ as we discussed for class number 1 and, in fact, (24) remains valid. But the function

$$[P_0][-P_1] = \prod_{p \in P_0} \frac{1}{1 - 1/p^s} \prod_{p \in P_1} \frac{1}{1 + 1/p^s},$$

on the contrary, is analytic at $s = 1$ owing to the fact, proven by Landau [14], that the classes P_0 and P_1 are equinumerous. For example, for $n = 6$, the primes in P_0 are those where the *characters* $(2/p)$ and $(-3/p)$ are both $+1$, while those in P_1 have both characters equal to -1. From this we find that

$$L_{-2}(s)L_3(s) = [P_0]^2[-P_1]^2[Q^2].$$

Therefore the $[P_0][-P_1][Q^2]$ in the second term of (36) is given by

$$(L_{-2}(s)L_3(s)[Q^2])^{1/2},$$

TABLE 5
Populations, Class No. 2

x	$B_6(x)$	$B_{2,3}(x)$	$B_{10}(x)$	$B_{2,5}(x)$
2^0	1	0	1	0
2^1	1	1	1	1
2^2	2	2	2	1
2^3	4	4	2	4
2^4	8	7	7	5
2^5	13	14	10	11
2^6	24	23	20	20
2^7	42	42	36	36
2^8	76	76	65	65
2^9	140	139	118	119
2^{10}	257	258	221	218
2^{11}	483	482	409	412
2^{12}	907	907	776	770
2^{13}	1717	1717	1463	1466
2^{14}	3272	3269	2788	2784
2^{15}	6261	6257	5328	5322
2^{16}	12027	12020	10222	10226
2^{17}	23172	23171	19714	19691
2^{18}	44769	44762	38054	38048
2^{19}	86708	86683	73685	73665
2^{20}	168245	168233	142944	142927
2^{21}	327073	327053	277838	277822
2^{22}	636849	636837	540889	540851
2^{23}	1241720	1241723	1054535	1054502
2^{24}	2424290	2424228	2058537	2058507
2^{25}	4738450	4738426	4023278	4023164

which is analytic at $s = 1$, and remains analytic for the real part of $s > \frac{1}{2}$ provided that the extended Riemann Hypothesis is true.

In any case, from the known zero-free regions of the L functions, the contribution of this second term in (36) cannot modify the counts $B_6(x)$ and $B_{2,3}(x)$ by more than $o(x \log^{-m} x)$ for any m. If $L_{-2}(s)$ and $L_3(s)$ satisfy the Riemann Hypothesis then this contribution is $o(x^{1/2+\epsilon})$ for any positive ϵ.

It follows in this case, and similarly in all 32 cases in Table 3, that (6) is again valid and that the formula for b_n in (27), with $\delta_n = 1$, must be altered only by the factor $\frac{1}{2}$ in (35):

$$b_n = \tfrac{1}{2}g_n[L_n(1) \cdot 2\,|\,n\,|/\pi\phi(2\,|\,n\,|)]^{1/2}. \tag{37}$$

We included in Table 1 values of b_n so computed for $n = -10, -7, -6, -3, 5, 6, 10$, and 13, but the last of these is rather crude since the constants $L_{13}(2^k)$ were not available. This difficulty, which was mentioned before, would also exist for $n = 22$, 37, and 58. (This problem is accentuated here since in these three cases g_n tends to differ from 1 by more than is usual. This is related to the fact that there are especially many primes of the form $m^2 + a$ for $a = 22, 37, 58$ [9, p. 326]. Specifically, $g_{13} \approx 1.085$ and g_a for the other values of a mentioned would be even larger. Whereas, on the contrary, if $m^2 + a$ has relatively few primes, such as for $a = 11, 14$, we have values particularly close to 1. Thus, $g_{11} \approx 1.019$ and $g_{14} \approx 1.009$.)

For $n = -14$, and generally for any n of types a_7 and a_6 in Table 3, one could compute $B_n(x)$ from $O_n(x)$ as in (29). This is so since 2 is in P_0' and $E_n(x) = B_n(x/2)$ as before. But for $n = 6$, say, one has the more intricate recurrences:

$$E_6(x) = B_{2,3}(x/2), \qquad E_{2,3}(x) = B_6(x/2). \tag{38}$$

To compute $B_6(x)$, therefore, $O_6(x)$ does not suffice. One must also investigate the second class, and compute $O_{2,3}(x)$ too, whether one wishes to, or not. (The cross-computation of $B_6(x)$ and $B_{2,3}(x)$ from columns of data $O_6(x)$ and $O_{2,3}(x)$ is an operation a little like lacing shoes.)

We include in Table 5 values of $B_6(x)$, $B_{2,3}(x)$, $B_{10}(x)$, and $B_{2,5}(x)$. One notes at once that the two classes are closely equinumerous for both $n = 6$ and $n = 10$. Algorithmicly speaking, the recurrence computation just described tends to bring this about, for if $B_6(x) > B_{2,3}(x)$, say, then $E_6(2x) < E_{2,3}(2x)$, and that tends to compensate for the previous excess.

The interesting investigation suggests itself to attempt to prove that all complex zeros of $L_{-2}(s)$ and $L_3(s)$ are of the form $a_i + it_i$, with $a_i < \theta < 1$ for some θ and all $t_i < T$, by utilizing the difference $B_6(x) - B_{2,3}(x)$ for all $x < X$. If this were successful, there would be a distinct gain, for the computation of this latter difference is very simple, while the direct investigation of the L functions requires elaborate transcendental calculations.

Before leaving this section we note that $B_n(x)(\log x)^{1/2}/b_n x$ converges to 1 somewhat more slowly for $n = 6$ and 10 than for $n = 1$. The data implies that c_6 (c_{10}) is about 34% (37%) larger than c_1.

Of perhaps greater interest is the following observation. If p is prime, it is well known that the period ρ of the regular continued fraction for $p^{1/2}$ is odd or even according as $p \equiv 1$ or $p \equiv -1 \pmod 4$. Further, the ρth convergent, P_ρ/Q_ρ, in the first case satisfies

$$P_\rho^2 - pQ_\rho^2 = -1. \tag{39}$$

But Table 4 calls attention to the less known result that if $p \equiv 3 \pmod 8$ as in type a_3, then $\rho = 4k + 2$ and

$$P_{2k+1}^2 - pQ_{2k+1}^2 = -2, \tag{40}$$

while if $p \equiv 7 \pmod 8$ as in type a_6, then $\rho = 4k$ and

$$P_{2k}^2 - pQ_{2k}^2 = +2. \tag{41}$$

8. Class Number 2 Continued, Other Square-Free n. Analogous to types (b) and (c) in Section 3 are 9 other cases of class number 2. These are summarized in Table 6. We spare the reader the details here. Suffice it to say that (6) is again valid, the two classes are again disjoint, and b_n is again computed by (27) with the additional factor of $\frac{1}{2}$. We have computed no examples here.

TABLE 6
Other Variations, Class No. 2

Type	n	F_0	F_1	P_0'	P_1'
b_1	-93	$u^2 - 93v^2$	$93u^2 - v^2$	31	3
b_1	-77	$u^2 - 77v^2$	$77u^2 - v^2$	11	7
b_1	-69	$u^2 - 69v^2$	$69u^2 - v^2$	3	23
b_1	-21	$u^2 - 21v^2$	$21u^2 - v^2$	7	3
b_2	-85	$u^2 - 85v^2$	$5u^2 - 17v^2$	—	5, 17
c_1	-57	$u^2 - 57v^2$	$57u^2 - v^2$	19	8, 3
c_1	-33	$u^2 - 33v^2$	$33u^2 - v^2$	3	8, 11
c_2	-65	$u^2 - 65v^2$	$5u^2 - 13v^2$	—	8, 5, 13
c_3	$+15$	$u^2 + 15v^2$	$3u^2 + 5v^2$	—	8, 3, 5

9. Class Number 2 with $n = 4n_0$; Splitting, Nonsplitting, and Overlapping Classes. Eight of the fourteen cases here derive from class number 1 cases $u^2 + n_0v^2$, namely: $n = -8, 8, -52, -20, 12, -68, 28$, and 16, where the corresponding n_0 cases have been discussed in Section 3. Five others derive from class number 2 cases in Table 3, namely: $n = -76, -44, -12, -92$, and -28. The interesting remaining case, $n = -32$, we will return to in the sequel.

We have already (inadvertently) come across the case $n = 12$. The form $u^2 + 3v^2$ represents 3 and primes of the form $6k + 1$. If v is even the prime is of the form $12k + 1$ while if v is odd the prime is either of the form $12k + 7$ or the single prime 3. If we confine ourselves to *odd numbers*, the two forms, $u^2 + 12v^2$ and $3u^2 + 4v^2$ behave precisely as in our previous class number 2 cases, including the facts that $O_{3,4}(x) \sim O_{12}(x)$ (as already noted on page 559), that the two classes are disjoint, that a formula such as (6) is again valid, as is also the relation

$$O_{3,4}(x) - O_{12}(x) = o\left(\frac{x}{(\log x)^m}\right) \quad \text{for any } m. \tag{42}$$

The *even* numbers represented by $u^2 + 12v^2$ and $3u^2 + 4v^2$, on the contrary, are identical, since in either case u must be even. Therefore, such a number equals $4(r^2 + 3s^2)$ for arbitrary r and s, and we have the equation

$$E_{12}(x) = E_{3,4}(x) = B_3(x/4) = E_3(x). \tag{43}$$

Since $O_3(x) \sim 3E_3(x)$, it easily follows that

$$B_{12}(x) \sim B_{3,4}(x) = \frac{b_{12}\,x}{(\log x)^{1/2}}\left[1 + \frac{c_{12}}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right] \tag{44}$$

where $b_{12} = \frac{5}{8}b_3$ as in Table 1.

Three other important cases here are $n = \pm 8$ and 16. These behave rather similarly to $n = 12$ except for the somewhat different behavior of the even numbers represented. This difference reflects the related difference concerning factors of 2 in the cases $n_0 = 3, \pm 2$, and 4.

Specifically, for $n_0 = \pm 2, 4$, we find that the primes $u^2 + n_0v^2$ *split* in two classes depending upon whether v is even or odd. Thus, we find that these primes, and also *all* odd numbers, fall into the following arithmetic progressions:

v	$n_0 = 4$	$n_0 = 2$	$n_0 = -2$
even	$8k + 1$	$8k + 1$	$8k + 1$
odd	$8k + 5$	$8k + 3$	$8k + 7$

Hence, the odd numbers given by the two forms

$$u^2 + 4n_0v^2, \qquad (2u + v)^2 + n_0v^2$$

again are disjoint, equinumerous, and have the same asymptotic behavior. We have written the second form here in such a way as to make it clear that if n_0 is even and $(2u + v)^2 + n_0v^2$ is odd, then $2u + v$ and v are also odd. In particular, for $n_0 = 4$, we have

$$(45) \qquad O_{16}(x) \sim O_{4,4,5}(x) \sim \tfrac{1}{2}O_4(x).$$

Again, as before, the two forms represent identical even numbers, and we have

$$(46) \qquad E_{4n_0}(x) = B_{n_0}(x/4)$$

in all these cases. But, for $n_0 = \pm 2$, $u^2 + 4n_0v^2$ now represents only one-half of the evens represented by $u^2 + n_0v^2$, while, for $n_0 = 4$, one may verify that $u^2 + 16v^2$ represents three-quarters of the even numbers of the form $u^2 + 4v^2$. Specifically, the evens of the form $u^2 + 16v^2$ comprise products of any odd number $u^2 + v^2$ times 4, 16, 32, 64, 128, or any higher power of 2. It is clear, in fact, that in the even numbers $u^2 + 4v^2$ we lose the even factor 2^1, in $u^2 + 16v^2$ we further lose 2^3, in $u^2 + 64v^2$ we further lose 2^5, etc.

In the cases $n = \pm 8$, 16 we therefore again find (6) valid with the relations

$$(47) \qquad b_{\pm 8} = \tfrac{1}{2}b_{\pm 2}, \qquad b_{16} = \tfrac{7}{16}b_1$$

as in Table 1. (The interested reader can also compute c_{16}.)

In Table 7 we have included data for $B_{12}(x)$ and $B_{16}(x)$ together with their related forms. We again record the redundancies:

$$(48) \qquad B_{12}(x) + B_{3,4}(x) = B_3(x) + B_3(x/4)$$

and the more complicated

$$(49) \quad B_{16}(x) + B_{4,4,5}(x) = B_1(x) - B_1\left(\frac{x}{2}\right) + 2B_1\left(\frac{x}{4}\right) - 2B_1\left(\frac{x}{8}\right) + 2B_1\left(\frac{x}{16}\right).$$

We also note that in Table 7, as in Table 5, the *principal* form

$$u^2 + nv^2$$

usually, but not always, leads its related form in population. This phenomenon is clearly somewhat similar to the Chebyshev phenomena, cf. [16], but we have not attempted to analyze it, even heuristically.

In contrast to the examples just discussed, consider $n = -12$. Here $n_0 = -3$ and we already have class number 2. We now find that the odd numbers of the form $u^2 - 12v^2$ include *all* of those of the form $u^2 - 3v^2$; there is no splitting. For v

TABLE 7
Populations, Other Class No. 2

x	$B_{12}(x)$	$B_{3,4}(x)$	$B_{16}(x)$	$B_{4,4,5}(x)$
2^0	1	0	1	0
2^1	1	0	1	0
2^2	2	2	2	1
2^3	2	3	2	2
2^4	6	5	4	4
2^5	9	9	8	7
2^6	17	16	13	14
2^7	30	29	25	24
2^8	54	53	44	43
2^9	98	98	83	82
2^{10}	183	181	152	149
2^{11}	341	341	286	284
2^{12}	645	640	538	534
2^{13}	1220	1218	1020	1015
2^{14}	2327	2321	1942	1937
2^{15}	4451	4449	3725	3713
2^{16}	8555	8546	7145	7136
2^{17}	16489	16482	13781	13759
2^{18}	31859	31845	26627	26597
2^{19}	61717	61707	51572	51537
2^{20}	119779	119760	100099	100045
2^{21}	232919	232865	194633	194586
2^{22}	453584	453511	379037	378987
2^{23}	884544	884493	739250	739161
2^{24}	1727213	1727125	1443573	1443465
2^{25}	3376505	3376376	2822186	2821923
2^{26}	6607371	6607207	5522889	5522689

even in $m = u^2 - 3v^2$, it is clear that m is represented. But for v odd, and therefore u even, we may utilize

$$m = u^2 - 3v^2 = (2u + 3v)^2 - 3(u + 2v)^2.$$

On the right $u + 2v$ is now even, so that m is still represented. For example:

$$13 = 4^2 - 3\cdot 1^2 = 11^2 - 3\cdot 6^2 = 11^2 - 12\cdot 3^2.$$

Therefore

$$O_{-12}(x) = O_{-3}(x)$$

and $n = -12$ still belongs to class number 2. As before we have

$$E_{-12}(x) = B_{-3}(x/4).$$

Whereas Gauss and his contemporaries studied quadratic forms for all n, there is some modern tendency to confine oneself to square-free n. We must note here, however, that we often find those n which are divisible by squares to be of special interest. Thus, we have seen, for the first time in this section, the phenomena of splitting, nonsplitting and overlapping classes. Further, as we shall see, while some class number 3, 4, or 8 cases, which are of much interest, are difficult to compute when they first occur at $n = 11, 14, 21, 41$, and 56, one can study essentially similar behavior for $n = 27, 20, 24, 256$, and 96, respectively, and now the computation goes through much more easily since the corresponding Dirichlet series $L_n(s)$ are known.

10. Class Number 2 Concluded. For brevity, and to allow us to proceed more quickly to essentially new phenomena, we forego a complete treatment of the previous cases. An interested reader can easily fill the gaps. For example, the fact that class number 2 cases $n_0 = -23, -19, -11$, and -7 remain class number 2 for $n = 4n_0$, just as it did for $n_0 = -3$, follows from the fact that if

$$u^2 - n_0 v^2 = \text{odd}, \qquad x^2 - n_0 y^2 = 1,$$

with v odd and u even, then

$$u^2 - n_0 v^2 = (xu + n_0 yv)^2 - n_0(yu + xv)^2, \tag{50}$$

and now $yu + xv$ is even since the fundamental solutions $x + y\sqrt{n_0}$ are given by $24 + 5\sqrt{23}$, $170 + 39\sqrt{19}$, $10 + 3\sqrt{11}$, and $8 + 3\sqrt{7}$, respectively. Similarly, the remaining nine cases of class number 2 in our range, namely those where n is divisible by an odd square: $n = 9, 18, -18, -27, -45, -54, 25, -50$, and 98, we also bypass. The reader will have no difficulty in verifying, for example, that $b_9 = 5b_1/9$ as in Table 1.

Since our paper is a long one, and its publication has been unduly delayed (the work was mostly done several years ago), we now declare an intermission. In the sequel we shall continue with the more intricate class number 4, 8, and 3 phenomena, and with discussion of some of the topics mentioned at the end of Section 1.

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